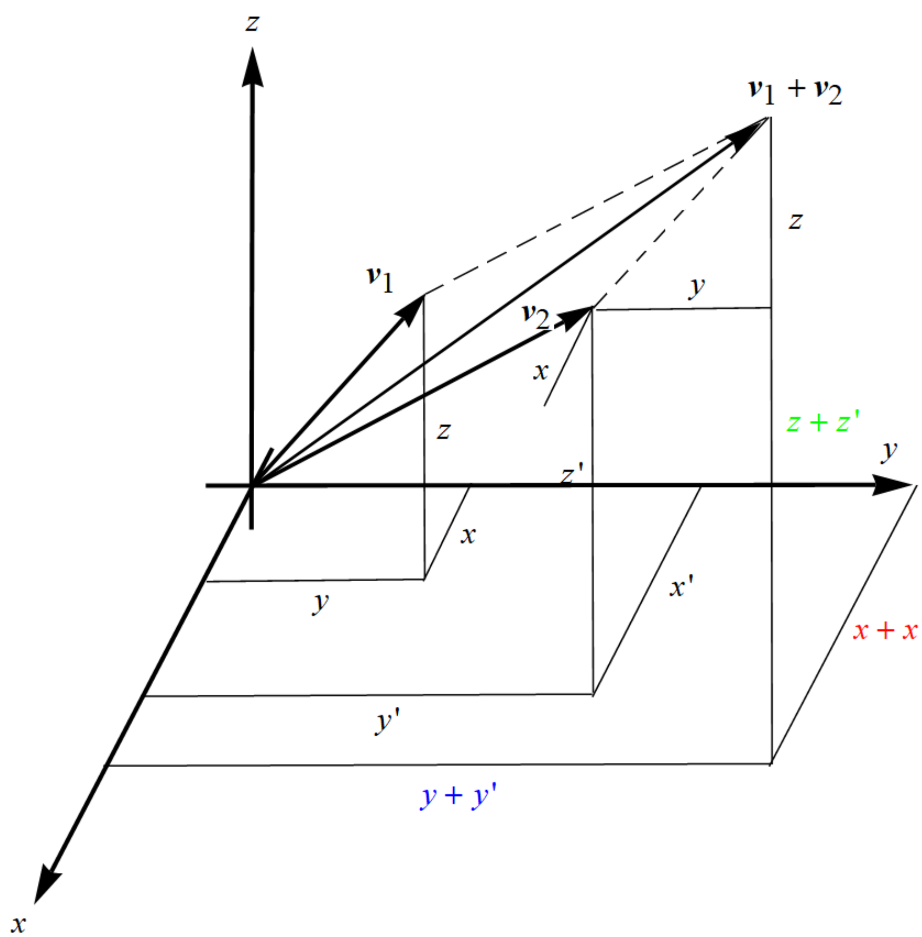
Exercise 12

(a) Generalize the geometric construction in Figure 1.1.7 to show that if $\mathbf{v}_1 = (x, y, z)$ and $\mathbf{v}_2 = (x', y', z')$, then $\mathbf{v}_1 + \mathbf{v}_2 = (x + x', y + y', z + z')$.

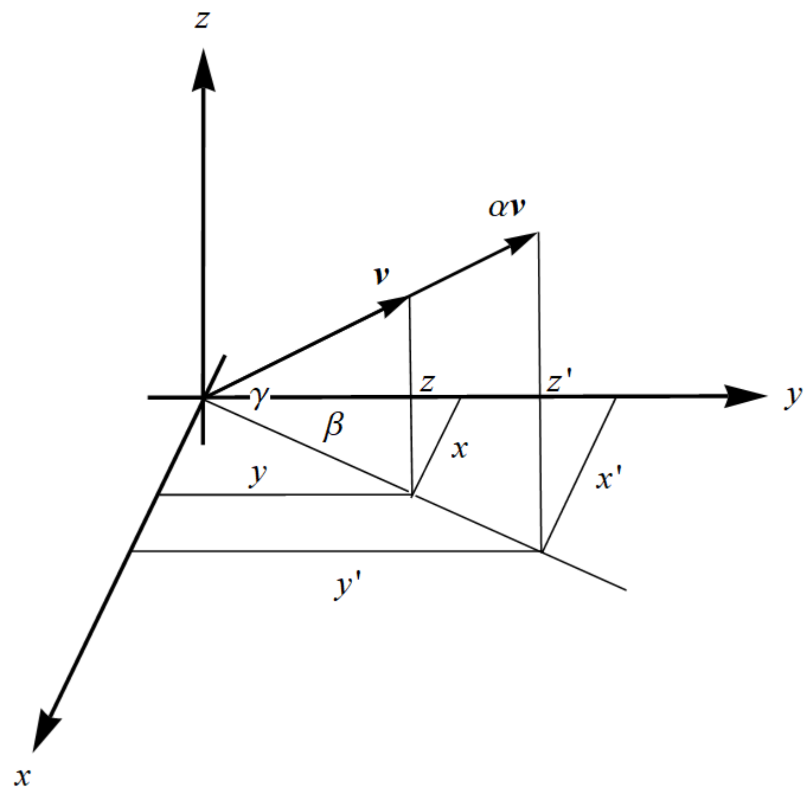
(b) Using an argument based on similar triangles, prove that $\alpha\mathbf{v} = (\alpha x, \alpha y, \alpha z)$ when $\mathbf{v} = (x, y, z)$.

Solution

Part (a)



Part (b)



Multiplying a vector by a number α changes its magnitude by a factor of α as shown in the figure above. By the Pythagorean theorem, the hypotenuses in the xy-plane have lengths, $\sqrt{x^2+y^2}$ and $\sqrt{x'^2+y'^2}$, respectively.

$$\sin\gamma = \frac{z}{|\mathbf{v}|} = \frac{z'}{|\alpha\mathbf{v}|} \quad \to \quad \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z'}{\alpha\sqrt{x^2+y^2+z^2}} \quad \to \quad z' = \alpha z$$

$$\tan\gamma = \frac{z}{\sqrt{x^2+y^2}} = \frac{z'}{\sqrt{x'^2+y'^2}} \quad \to \quad \frac{\sqrt{x'^2+y'^2}}{\sqrt{x^2+y^2}} = \frac{z'}{z} = \alpha$$

$$\cos\beta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y'}{\sqrt{x'^2+y'^2}} \quad \to \quad \frac{\sqrt{x'^2+y'^2}}{\sqrt{x^2+y^2}} = \alpha = \frac{y'}{y} \quad \to \quad y' = \alpha y$$

$$\sin\beta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x'}{\sqrt{x'^2+y'^2}} \quad \to \quad \frac{\sqrt{x'^2+y'^2}}{\sqrt{x^2+y^2}} = \alpha = \frac{x'}{x} \quad \to \quad x' = \alpha x$$

Therefore, $\alpha\mathbf{v} = (x', y', z') = (\alpha x, \alpha y, \alpha z)$.